

SELFEXTENSIONAL LOGICS WITH A NEARLATTICE TERM

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ABSTRACT. In this paper, we define when a ternary connective m of an algebraic language \mathcal{L} is called a DN-term of a sentential logic \mathcal{S} . We characterize the selfextensional logics with a DN-term as those logics that can be defined using the order induced by the interpretation of the nearlattice operation in the algebras of their algebraic counterpart. We prove that the canonical class of algebras associated with a selfextensional logic with a DN-term is a variety and we thus obtain that the logic is in fact fully selfextensional.

1. INTRODUCTION

This paper is motivated by the results and ideas given in [8] and [9]. In [8], selfextensional logics with a binary term \rightarrow satisfying the deduction-detachment theorem are studied proving that they can be characterized as the logics \mathcal{S} for which there is a class of algebras K such that the equations that define Hilbert algebras hold for \rightarrow and the following condition is satisfied:

$$\varphi_0, \dots, \varphi_{n-1} \vdash_{\mathcal{S}} \varphi \iff (\forall A \in K)(\forall h \in \text{Hom}(Fm, A)) \\ h(\varphi_0 \rightarrow (\dots \rightarrow (\varphi_{n-1} \rightarrow \varphi) \dots)) = 1.$$

Similar results are obtained in [9]. There, selfextensional logics with a conjunction \wedge are characterized as the logics \mathcal{S} for which there is a class of algebras K such that the semilattice equations are satisfied for \wedge and the following condition holds:

$$\varphi_0, \dots, \varphi_{n-1} \vdash_{\mathcal{S}} \varphi \iff (\forall A \in K)(\forall h \in \text{Hom}(Fm, A)) \\ h(\varphi_0) \wedge \dots \wedge h(\varphi_{n-1}) \leq h(\varphi).$$

The notion of distributive nearlattice can be defined in two equivalent way. An algebra $\langle A, m \rangle$ of type (3) is a distributive nearlattice if satisfies some identities (see Definition 2.5 and Proposition 2.8); and a ternary algebra $\langle A, m \rangle$ is a distributive nearlattice if and only if A with the binary term $x \vee y := m(x, x, y)$ is a join-semilattice such that for every $a \in A$, the upset $[a]$ is a distributive lattice with respect to the order induced by \vee .

Hence, on the one hand, we can consider the distributive nearlattices as a generalization of the variety of Tarski algebras. As it is known, a

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Tarski algebra A can be defined as a binary algebra $\langle A, \rightarrow \rangle$ satisfying some identities and equivalently can be defined as a join-semilattice $\langle A, \vee \rangle$ such that for every $a \in A$, the upset $[a]$ is a Boolean algebra with respect to the order induced by \vee . On the other hand, we may consider the distributive nearlattices very close to the distributive lattices. Because, although the meet of two arbitrary elements in a distributive nearlattice may not exist, the meet of two elements x and y exists whenever x and y have a lower bound.

The aim of this paper is to propose a definition of when a ternary term m of an algebraic language \mathcal{L} is a distributive nearlattice term for a sentential logic \mathcal{S} . We show that selfextensional logics with a distributive nearlattice term m can be characterized as the logics \mathcal{S} for which there exists a class of algebras \mathbf{K} such that the $\{m\}$ -reducts of the algebras of \mathbf{K} are distributive nearlattices and the consequence relation of \mathcal{S} can be defined using the order induced by the term m on the algebras of \mathbf{K} .

2. PRELIMINARIES

In the first part of this section we introduce some concepts and results of Abstract Algebraic Logic (AAL) needed to follow the paper and in the second part of the section, we present the algebraic theory of nearlattices. Our main references for AAL are [3, 4, 5] and for the theory of nearlattice are [2, 7, 1].

2.1. Abstract Algebraic Logic. Let \mathcal{L} be an algebraic language (or algebraic similarity type). We denote by $Fm(\mathcal{L})$ the absolutely free algebra of type \mathcal{L} with a denumerable set Var of propositional variables as the set of generators. The algebra $Fm(\mathcal{L})$ is called the *algebra of formulas* of type \mathcal{L} and its elements are called *formulas*. When there is no danger of confusion, we write Fm instead of $Fm(\mathcal{L})$.

A *sentential logic* (also called *deductive system* in AAL) of type \mathcal{L} is a pair $\mathcal{S} = \langle Fm, \vdash_{\mathcal{S}} \rangle$ where Fm is the algebra of formulas of type \mathcal{L} and $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(Fm) \times Fm$ is a relation satisfying the following properties: for all $\Gamma, \Delta \subseteq Fm$ and $\varphi \in Fm$ (as usual we write $\Gamma \vdash_{\mathcal{S}} \varphi$ for $(\Gamma, \varphi) \in \vdash_{\mathcal{S}}$),

- (S1) if $\varphi \in \Gamma$ then $\Gamma \vdash_{\mathcal{S}} \varphi$;
- (S2) if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathcal{S}} \varphi$;
- (S3) if $\Gamma \vdash_{\mathcal{S}} \varphi$ and for every $\gamma \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \gamma$ then $\Delta \vdash_{\mathcal{S}} \varphi$;
- (S4) if $\Gamma \vdash_{\mathcal{S}} \varphi$ then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathcal{S}} \varphi$;
- (S5) if $\Gamma \vdash_{\mathcal{S}} \varphi$ then $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$ for all substitution $\sigma \in \text{Hom}(Fm, Fm)$.

The relation $\vdash_{\mathcal{S}}$ is called the *consequence relation* of \mathcal{S} . A set $\Gamma \subseteq Fm$ is called a *theory* of \mathcal{S} (\mathcal{S} -*theory*, for short) if is closed under the consequence relation of \mathcal{S} , that is, for every formula $\varphi \in Fm$, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\varphi \in \Gamma$. Let us denote by $\text{Th}(\mathcal{S})$ the collection of all \mathcal{S} -theories. It is easy to see that $\text{Th}(\mathcal{S})$ is an algebraic closure system on Fm and the closure operator

associated with $\text{Th}(\mathcal{S})$, which is denoted by $C_{\mathcal{S}}$, is defined as:

$$\varphi \in C_{\mathcal{S}}(\Gamma) \iff \Gamma \vdash_{\mathcal{S}} \varphi$$

for all $\Gamma \cup \{\varphi\} \subseteq Fm$. Moreover, it is clear that $C_{\mathcal{S}}$ is finitary.

Let \mathcal{S} be a sentential logic. The *Frege relation* of \mathcal{S} , in symbols $\Lambda(\mathcal{S})$, is the interderivability relation. That is, $(\varphi, \psi) \in \Lambda(\mathcal{S})$ if and only if $\varphi \vdash_{\mathcal{S}} \psi$ and $\psi \vdash_{\mathcal{S}} \varphi$. The Frege relation of a sentential logic is an equivalence relation but it is not necessarily a congruence on Fm . A sentential logic \mathcal{S} is said to be *selfextensional* if the Frege relation $\Lambda(\mathcal{S})$ is a congruence on Fm .

Let A be an algebra of the same similarity type as \mathcal{S} . A subset $F \subseteq A$ is said to be an \mathcal{S} -*filter* of A if and only if for any $\Gamma \cup \{\varphi\} \subseteq Fm$ and any interpretation $h \in \text{Hom}(Fm, A)$,

$$\text{if } \Gamma \vdash_{\mathcal{S}} \varphi \text{ and } h[\Gamma] \subseteq F \text{ then } h(\varphi) \in F.$$

The set of all \mathcal{S} -filters on a given algebra A is denoted by $\text{Fis}_{\mathcal{S}}(A)$; this set is an algebraic closure system. The associated closure operator will be denoted by $\text{Fis}_{\mathcal{S}}^A$.

Let \mathcal{L} be a fixed but arbitrary algebraic language. A *generalized matrix*, *g-matrix* for short, of similarity type \mathcal{L} is a pair $\langle A, C \rangle$ where A is an algebra of type \mathcal{L} and C is an algebraic closure system on A . We denote by C the closure operator associated with C and we will often identify the g-matrix $\langle A, C \rangle$ with the pair $\langle A, C \rangle$. Notice that the closure operator C is finitary, i.e., for all $X \cup \{a\} \subseteq A$, $a \in C(X)$ implies that there is a finite $X_0 \subseteq X$ such that $a \in C(X_0)$. The reader should be kept in mind that all logics and g-matrices considered in this paper are finitary and thus some general results of AAL are restricted to this assumptions.

One of the most interesting aspects of g-matrices is that they can be used in a completely natural way as models of sentential logics and as models of Gentzen systems. This double function of g-matrices allows relating the algebraic theory of sentential logic to the Gentzen systems. We direct the interested reader on these topics to [4] and [5].

An important example of g-matrix is given by a sentential logic \mathcal{S} . If \mathcal{S} is a sentential logic, then $\langle Fm, \text{Th}(\mathcal{S}) \rangle$ is a g-matrix.

Definition 2.1. A g-matrix $\langle A, C \rangle$ is said to be a *g-model* of a sentential logic \mathcal{S} when for all $\Gamma \cup \{\varphi\} \subseteq Fm$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ then $h(\varphi) \in C(h[\Gamma])$ for all $h \in \text{Hom}(Fm, A)$. Let us denote the class of all g-models of a sentential logic \mathcal{S} by $\mathbf{GMod}(\mathcal{S})$.

The logical concept of Frege relation is extended to the setting of g-matrices. The *Frege relation* of a g-matrix $\langle A, C \rangle$ is defined by:

$$(a, b) \in \Lambda_A(C) \iff C(a) = C(b)$$

for every $a, b \in A$. The *Tarski congruence* of a g-matrix $\langle A, C \rangle$ is the largest congruence below the Frege relation of the g-matrix. We denote the Tarski congruence of $\langle A, C \rangle$ by $\Omega_A(C)$. A g-matrix is said to be *reduced* when its

Tarski congruence is the identity relation. Let us denote by $\mathbf{GMod}^*(\mathcal{S})$ the class of all reduced g-models of a sentential logic \mathcal{S} .

We can now introduce the class of algebras that is considered in AAL as the algebraic counterpart of a sentential logic.

Definition 2.2. The *canonical class of algebras* associated with a sentential logic \mathcal{S} is the class of the algebraic reducts of the reduced g-models of \mathcal{S} ; it is denoted by $\text{Alg}(\mathcal{S})$. That is,

$$\begin{aligned} \text{Alg}(\mathcal{S}) &:= \text{Alg}(\mathbf{GMod}^*(\mathcal{S})) \\ &= \{A : \langle A, C \rangle \in \mathbf{GMod}^*(\mathcal{S}) \text{ for some finitary closure operator } C\}. \end{aligned}$$

Moreover, another important class of algebras associated to \mathcal{S} is $K_{\mathcal{S}} := \mathbb{V}(Fm/\tilde{\Omega}(\mathcal{S}))$, the variety generated by the algebra $Fm/\tilde{\Omega}(\mathcal{S})$. This variety is called *the intrinsic variety of \mathcal{S}* .

Now we present some known relations between the previous classes of algebras.

Lemma 2.3. *Let \mathcal{S} be a sentential logic. Then, the intrinsic variety of \mathcal{S} is the variety generated by the class $\text{Alg}(\mathcal{S})$ and hence we have $\text{Alg}(\mathcal{S}) \subseteq \mathbb{V}(\text{Alg}(\mathcal{S})) = K_{\mathcal{S}}$.*

The notion of *fully selfextensional* logic has several useful characterizations.

Definition 2.4. A sentential logic \mathcal{S} is called *fully selfextensional* if one of the following equivalent conditions hold:

- (1) for every algebra A , the Frege relation of the g-matrix $\langle A, \text{Fi}_{\mathcal{S}}(A) \rangle$ is a congruence on A ;
- (2) for every $A \in \text{Alg}(\mathcal{S})$, the Frege relation of the g-matrix $\langle A, \text{Fi}_{\mathcal{S}}(A) \rangle$ is the identity relation.

2.2. Nearlattices. The notion of nearlattice can be presented in two different and equivalent ways. They can be defined as join-semilattices that satisfy some property and can be defined as algebras with only one ternary connective satisfying some identities. The two different ways to consider nearlattices are useful to different purposes.

Definition 2.5. An algebra $\langle A, m \rangle$ of type (3) is called a *nearlattice* if the following identities hold:

- (P1) $m(x, y, x) = x$,
- (P2) $m(x, x, y) = m(y, y, x)$,
- (P3) $m(m(x, x, y), m(x, x, y), z) = m(x, x, m(y, y, z))$,
- (P4) $m(x, y, z) = m(y, x, z)$,
- (P5) $m(m(x, y, z), w, z) = m(x, m(y, w, z), z)$,
- (P6) $m(x, m(y, y, x), z) = m(x, x, z)$,
- (P7) $m(x, x, m(x, y, z)) = m(x, x, z)$,
- (P8) $m(m(x, x, z), m(y, y, z), z) = m(x, y, z)$.

Theorem 2.6. (1) If $\langle A, m \rangle$ is a nearlattice, then the algebra $A_* = \langle A, \vee \rangle$, where

$$(2.1) \quad x \vee y := m(x, x, y),$$

is a join-semilattice such that for every $a \in A$ the principal upset $[a] = \{b \in A : a \leq b\}$ is a lattice with respect to the order induced by \vee .

(2) If $\langle S, \vee \rangle$ is a join-semilattice such that every principal upset is a lattice, then the algebra $S^* = \langle S, m \rangle$ with

$$m(x, y, z) := (x \vee z) \wedge_z (y \vee z).$$

is a nearlattice.

(3) If A is a nearlattice and S is join-semilattice such that every principal upset is a lattice, then $(A_*)^* = A$ and $(S^*)_* = S$.

This theorem shows us that there is a one-to-one correspondence between nearlattices and join-semilattices where every principal upset is a lattice. Hence, the join-semilattices where all principal upsets are lattices can also be called *nearlattices*. We will consider nearlattices as ternary algebras $\langle A, m \rangle$ satisfying the identities (P1)-(P8) and we consider the join operation \vee on A defined as in (2.1). Moreover, the partial order \leq on A is determined by \vee , i.e., $x \leq y$ if and only if $y = x \vee y = m(x, x, y)$.

Let A be a nearlattice. For every element $a \in A$, we denote the meet in $[a]$ by \wedge_a . It should be noted that the meet $x \wedge y$ exists in A if and only if x, y have a common lower bound in A . Thus, sometimes we shall write $x \wedge y$ instead of $x \wedge_a y$ for elements $x, y \in [a]$. This should be kept in mind since we will use it without mention.

Definition 2.7. A nearlattice $\langle A, m \rangle$ is called *distributive* if for every $a \in A$, the lattice $\langle [a], \vee, \wedge_a \rangle$ is distributive.

Proposition 2.8. A nearlattice $\langle A, m \rangle$ is distributive if and only if it satisfies either of the following (equivalent) identities:

$$(D1) \quad m(x, m(y, y, z), w) = m(m(x, y, w), m(x, y, w), m(x, z, w));$$

$$(D2) \quad m(x, x, m(y, z, w)) = m(m(x, x, y), m(x, x, z), w).$$

We denote by \mathbb{DN} the variety of all distributive nearlattices. Let $\mathbf{2} = \{0, 1\}$ be the *two-element* distributive nearlattice with $0 < 1$. Then, it can be proved that \mathbb{DN} is the variety generated by $\mathbf{2}$, i.e., $\mathbb{DN} = \mathbb{V}(\mathbf{2})$ ([2, Corollary 2.7.6]).

Definition 2.9. Let $\langle A, m \rangle$ be a nearlattice. Let $I, F \subseteq A$ be nonempty.

- (1) I is said to be an *ideal* of A if
 - (i) $y \in I$ and $x \leq y$ implies $x \in I$,
 - (ii) if $x, y \in I$, then $x \vee y \in I$.
- (2) F is said to be a *filter* of A if
 - (i) $x \in F$ and $x \leq y$ implies $y \in F$,
 - (ii) if $x, y \in F$ and $x \wedge y$ exists in A , then $x \wedge y \in F$.

Let us denote by $\text{Fi}(A)$ the collection of all filters of a nearlattice A . It is easy to check that for every nearlattice A the intersection of any collection of filters is either a filter or an empty set. So, for every nonempty $X \subseteq A$, there exists the least filter containing X ; it is denoted by $\text{Fi}_A(X)$. If $X = \{a_1, \dots, a_n\}$, then we write $\text{Fi}_A(a_1, \dots, a_n)$ instead $\text{Fi}_A(\{a_1, \dots, a_n\})$: it is easy to check that $\text{Fi}_A(a) = [a]$.

Proposition 2.10. *Let A be a nearlattice and $F \subseteq A$ be nonempty. Then, the following conditions are equivalent:*

- (1) $F \in \text{Fi}(A)$;
- (2) if $a, b \in F$, then $m(a, b, c) \in F$ for all $c \in A$.

Proof. (1) \Rightarrow (2) Let $a, b \in F$ and $c \in A$. As F is an upset, $a \vee c, b \vee c \in F$ and since F is closed under existing meets, it follows that $m(a, b, c) = (a \vee c) \wedge (b \vee c) \in F$.

(2) \Rightarrow (1) Let $a \in F$ and $b \in A$ be such that $a \leq b$. So $b = a \vee b = m(a, a, b) \in F$. Thus F is an upset. Let $a, b \in F$ be such that $a \wedge b$ exists. By (2), we obtain that $a \wedge b = m(a, b, a \wedge b) \in F$. Hence $F \in \text{Fi}(A)$. \square

Now we introduce the following definition that will be useful for what follows.

Definition 2.11. Let $\langle A, m \rangle$ be a nearlattice. For each natural number n we define inductively, for all $a_1, \dots, a_n, b \in A$, the element $m^{n-1}(a_1, \dots, a_n, b)$ as follows:

- $m^0(a_1, b) := m(a_1, a_1, b)$ and
- for $n > 1$, $m^{n-1}(a_1, \dots, a_n, b) := m(m^{n-2}(a_1, \dots, a_{n-1}, b), a_n, b)$.

Thus $m^0(a_1, b) = a_1 \vee b$ and $m^1(a_1, a_2, b) = m(a_1, a_2, b)$. The following proposition follows directly by induction and thus we omit its proof.

Proposition 2.12. *Let $\langle A, m_A \rangle$ and $\langle B, m_B \rangle$ be nearlattices and let $h \in \text{Hom}(A, B)$. Then $h(m_A^{n-1}(a_1, \dots, a_n, b)) = m_B^{n-1}(h(a_1), \dots, h(a_n), h(b))$ for all $a_1, \dots, a_n, b \in A$.*

The proofs of the following two propositions can be found in [6].

Proposition 2.13. *Let $\langle A, m \rangle$ be a nearlattice and $a_1, \dots, a_n, a_{n+1}, a, b \in A$. Then:*

- (1) $m^{n-1}(a_1, \dots, a_n, b) = (a_1 \vee b) \wedge_b \dots \wedge_b (a_n \vee b)$;
- (2) $b \leq m^{n-1}(a_1, \dots, a_n, b)$;
- (3) if $a \leq a_i$ for all $i \in \{1, \dots, n\}$, then $a \leq m^{n-1}(a_1, \dots, a_n, b)$;
- (4) $m^n(a_1, \dots, a_{n+1}, b) \leq m^{n-1}(a_1, \dots, a_n, b)$;
- (5) $m^{n-1}(a_1, \dots, a_n, b) = m^{n-1}(a_{\sigma(1)}, \dots, a_{\sigma(n)}, b)$ for every permutation σ of $\{1, \dots, n\}$.

In the following proposition, we establish a characterization of the distributivity condition on nearlattice.

Proposition 2.14. *Let A be a nearlattice. Then the following conditions are equivalent:*

- (1) *A is distributive;*
- (2) *for all $a, a_1, \dots, a_n \in A$,*

$$a \in \text{Fi}_A(a_1, \dots, a_n) \text{ implies } a \in \text{Fi}_A(m^{n-1}(a_1, \dots, a_n, a)).$$

Remark 2.15. Let A be a distributive nearlattice and let $a_1, \dots, a_n, a \in A$. From the previous two propositions we have

$$a \in \text{Fi}_A(a_1, \dots, a_n) \iff a = m^{n-1}(a_1, \dots, a_n, a).$$

3. DISTRIBUTIVE NEARLATTICE-BASED LOGICS

Definition 3.1. A class of algebras K of a given similarity type \mathcal{L} is called *distributive nearlattice-based* (DN-based for short) if there is a ternary term m such that the distributive nearlattice equations (P1)-(P8) (page 4) and (D1) (page 5) hold in K . In this case we will also say that K is a DN-class relative to m and when there is not danger of confusion we simply say that K is a DN-class.

For a ternary term m of an algebraic language \mathcal{L} , we will consider the binary term \vee defined as $x \vee y := m(x, x, y)$. We also defined, for every natural number n and variables x_1, \dots, x_n, x , the formula $m^{n-1}(x_1, \dots, x_n, x)$ as follows:

- $m^0(x_1, x) := m(x_1, x_1, x)$
- for $n > 1$, $m^{n-1}(x_1, \dots, x_n, x) = m(m^{n-2}(x_1, \dots, x_{n-1}, x), x_n, x)$.

Notice that if K is a DN-class relative to m , then for every algebra $A \in K$ the $\{m\}$ -reduct $\langle A, m^A \rangle$ is a distributive nearlattice. Moreover, we have that the variety $\mathbb{V}(K)$ generated by a DN-class K is also a DN-class.

Now we introduce one of the main definitions of this section.

Definition 3.2. A sentential logic \mathcal{S} of type \mathcal{L} is said to be *distributive nearlattice-based* (DN-based for short) if and only if there is a ternary term m and a class of algebras K of type \mathcal{L} which is a DN-class relative to m and it holds that for every $n > 0$ and for all formulas $\varphi_1, \dots, \varphi_n, \varphi$,

$$(3.1) \quad \varphi_1, \dots, \varphi_n \vdash_{\mathcal{S}} \varphi \iff (\forall A \in K)(\forall h \in \text{Hom}(Fm, A)) \\ m^{n-1}(h(\varphi_1), \dots, h(\varphi_n), h(\varphi)) \leq h(\varphi).$$

We will say that \mathcal{S} is DN-based relative to m and K .

It should be noted, by property (5) of Proposition 2.13, that the order in which the formulas in (3.1) are taken is independent.

Let \mathcal{S} be a DN-based logic relative to m and K . It is easy to check that for every formulas φ and ψ ,

$$\varphi \vdash_{\mathcal{S}} \psi \iff (\forall A \in K)(\forall h \in \text{Hom}(Fm, A))(h(\varphi) \leq h(\psi)).$$

Then, we obtain that for all $\varphi, \psi \in Fm$

$$(3.2) \quad \varphi \Vdash_{\mathcal{S}} \psi \iff K \models \varphi \approx \psi \iff \mathbb{V}(K) \models \varphi \approx \psi.$$

Noticed that (3.2) is independent of the term m .

From (3.1) and by property (2) of Proposition 2.13, we have that

$$\begin{aligned}
\varphi_1, \dots, \varphi_n \vdash_{\mathcal{S}} \varphi &\iff (\forall A \in \mathbf{K})(\forall h \in \text{Hom}(Fm, A)) \\
&\quad m^{n-1}(h(\varphi_1), \dots, h(\varphi_n), h(\varphi)) \leq h(\varphi) \\
&\iff (\forall A \in \mathbf{K})(\forall h \in \text{Hom}(Fm, A)) \\
&\quad m^{n-1}(h(\varphi_1), \dots, h(\varphi_n), h(\varphi)) = h(\varphi) \\
&\iff \mathbf{K} \models m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \approx \varphi \\
&\iff \mathbb{V}(\mathbf{K}) \models m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \approx \varphi.
\end{aligned}$$

Hence, \mathcal{S} is also DN-based relative to the variety $\mathbb{V}(\mathbf{K})$ generated by \mathbf{K} . Moreover, by (3.2), we can see that the variety for which \mathcal{S} is DN-based is unique. So, let us denote the variety relative for which \mathcal{S} is DN-based by $\mathbb{V}(\mathcal{S})$.

Now we show some logical properties of the DN-based logics. The reader can compare these properties with the Gentzen-style rules introduced in Definition 4.2 to define the Gentzen system $\mathcal{G}_{\mathbb{DN}}$ associated with the variety of the distributive nearlattices \mathbb{DN} .

Proposition 3.3. *Let \mathcal{S} be a DN-based logic relative to m . Then, for all $\varphi, \psi, \chi \in Fm$, the following properties hold:*

- (B1) $\varphi \vee \psi \vdash_{\mathcal{S}} \chi$ if and only if $\varphi \vdash_{\mathcal{S}} \chi$ and $\psi \vdash_{\mathcal{S}} \chi$;
- (B2) $m(\varphi, \psi, \chi) \vdash_{\mathcal{S}} \varphi \vee \chi$ and $m(\varphi, \psi, \chi) \vdash_{\mathcal{S}} \psi \vee \chi$;
- (B3) $\varphi \vee \chi, \psi \vee \chi \vdash_{\mathcal{S}} m(\varphi, \psi, \chi)$;
- (B4) if $\varphi_1, \dots, \varphi_n \vdash_{\mathcal{S}} \varphi$, then $m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \vdash_{\mathcal{S}} \varphi$.

Proof. We have that \mathcal{S} is DN-based relative to $\mathbb{V}(\mathcal{S})$ and the ternary term m . Property (B1) is a consequence of the fact that for every $A \in \mathbb{V}(\mathcal{S})$, the $\{\vee\}$ -reduct $\langle A, \vee^A \rangle$ is a join-semilattice. Property (B2) holds, because for every $A \in \mathbb{V}(\mathcal{S})$ and all $a, b, c \in A$, we have $m^A(a, b, c) = (a \vee c) \wedge (b \vee c) \leq a \vee c, b \vee c$. In order to prove (B3), let $A \in \mathbb{V}(\mathcal{S})$ and $h \in \text{Hom}(Fm, A)$. Let $h(\varphi) = a$, $h(\psi) = b$ and $h(\chi) = c$. So, we need to show that $m(a \vee c, b \vee c, m(a, b, c)) \leq m(a, b, c)$. Now, by condition (D2), we have

$$\begin{aligned}
m(a \vee c, b \vee c, m(a, b, c)) &= c \vee m(a, b, m(a, b, c)) = m(a, b, m(a, b, c)) \\
&= (a \vee m(a, b, c)) \wedge (b \vee m(a, b, c)) = (a \vee c) \wedge (b \vee c) = m(a, b, c).
\end{aligned}$$

Hence (B3) holds. Lastly, property (B4) is an immediate consequence by (3.1). \square

Proposition 3.4. *Let \mathcal{S} be a DN-based logic relative to m . Then \mathcal{S} is selfextensional and $\mathbb{V}(\mathcal{S}) = \mathbf{K}_{\mathcal{S}}$.*

Proof. By definition of Frege relation and (3.2), we have $(\varphi, \psi) \in \Lambda(\mathcal{S}) \iff \mathbb{V}(\mathcal{S}) \models \varphi \approx \psi$ and hence we obtain that $\Lambda(\mathcal{S})$ is a congruence on Fm . Therefore \mathcal{S} is selfextensional. Now, since \mathcal{S} selfextensional, it follows that

$\varphi \dashv\vdash_{\mathcal{S}} \psi \iff K_{\mathcal{S}} \models \varphi \approx \psi$. Then, by (3.2) again, we obtain that $\mathbb{V}(\mathcal{S}) \models \varphi \approx \psi \iff \varphi \dashv\vdash_{\mathcal{S}} \psi \iff K_{\mathcal{S}} \models \varphi \approx \psi$ and therefore $\mathbb{V}(\mathcal{S}) = K_{\mathcal{S}}$. \square

Now we introduce the second important definition of the paper.

Definition 3.5. Let \mathcal{S} be a sentential logic over an algebraic language \mathcal{L} . A ternary term m of \mathcal{L} is said to be a *DN-term* of \mathcal{S} if and only if \mathcal{S} satisfies the properties (B1)-(B4) with regard to m .

Proposition 3.6. *If m is a DN-term of a sentential logic \mathcal{S} , then the following properties holds:*

- (1) $m^{n-1}(\varphi_1, \dots, \varphi_n, \psi) \vdash_{\mathcal{S}} \varphi_i \vee \psi$, for all $i \in \{1, \dots, n\}$;
- (2) $\varphi_1 \vee \psi, \dots, \varphi_n \vee \psi \vdash_{\mathcal{S}} m^{n-1}(\varphi_1, \dots, \varphi_n, \psi)$;
- (3) if $m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \vdash_{\mathcal{S}} \varphi$ then $\varphi_1, \dots, \varphi_n \vdash_{\mathcal{S}} \varphi$;
- (4) $\varphi \vdash_{\mathcal{S}} m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi)$.

Proof. Properties (1) and (2) can be proved by induction on n and from properties (B2) and (B3), respectively. Properties (3) and (4) are consequences from (B1) and (2). \square

Proposition 3.7. *Let \mathcal{S} be a sentential logic. If m and m' are DN-terms of \mathcal{S} , then $m(\varphi, \psi, \chi) \dashv\vdash_{\mathcal{S}} m'(\varphi, \psi, \chi)$, for all $\varphi, \psi, \chi \in Fm$.*

Proof. Let $\varphi, \psi, \chi \in Fm$. Since $\varphi \vdash_{\mathcal{S}} \varphi \vee' \psi$ and $\psi \vdash_{\mathcal{S}} \varphi \vee' \psi$, it follows by (B1) that $\varphi \vee \psi \vdash_{\mathcal{S}} \varphi \vee' \psi$. Similarly, we have $\varphi \vee' \psi \vdash_{\mathcal{S}} \varphi \vee \psi$ and hence $\varphi \vee \psi \dashv\vdash_{\mathcal{S}} \varphi \vee' \psi$. Now, using this and by (B2) and (B3), and some abuse of notation, we obtain

$$m(\varphi, \psi, \chi) \dashv\vdash_{\mathcal{S}} \varphi \vee \chi, \psi \vee \chi \dashv\vdash_{\mathcal{S}} \varphi \vee' \chi, \psi \vee' \chi \dashv\vdash_{\mathcal{S}} m'(\varphi, \psi, \chi). \quad \square$$

Thus we obtain that if \mathcal{S} is a DN-based logic relative to ternary terms m and m' , then for every $A \in \mathbb{V}(\mathcal{S})$ we have by the previous proposition and (3.2) that the ternary operations m^A and m'^A coincide. Hence, we can say simply that a logic \mathcal{S} is DN-based.

Now we are ready to show one of the main results of this paper.

Theorem 3.8. *Let \mathcal{S} be a sentential logic and m be a ternary term of \mathcal{S} . Then, \mathcal{S} is selfextensional and m is a DN-term of \mathcal{S} if and only if \mathcal{S} is a DN-based logic relative to m .*

Proof. The implication from right to left was proved in Propositions 3.3 and 3.4. Now we assume that \mathcal{S} is selfextensional and m is a DN-term of \mathcal{S} . First, since \mathcal{S} is selfextensional, it follows that $\Lambda(\mathcal{S})$ is a congruence on Fm and hence we can consider the quotient algebra $Fm^* := Fm/\Lambda(\mathcal{S})$. Let us show that $\langle Fm^*, m^* \rangle$, with $m^*(\overline{\varphi}, \overline{\psi}, \overline{\chi}) := \overline{m(\varphi, \psi, \chi)}$, is a distributive nearlattice. By (B1), \mathcal{S} satisfies the following properties: $\varphi \vee \varphi \vdash_{\mathcal{S}} \varphi$, $\varphi \vdash_{\mathcal{S}} \varphi \vee \psi$, $\varphi \vee \psi \vdash_{\mathcal{S}} \psi \vee \varphi$ and $\varphi \vee (\psi \vee \chi) \dashv\vdash_{\mathcal{S}} (\varphi \vee \psi) \vee \chi$. Thus, it is easy to check that $\langle Fm^*, \vee^* \rangle$, with $\overline{\varphi} \vee^* \overline{\psi} := m^*(\overline{\varphi}, \overline{\varphi}, \overline{\psi})$, is a join-semilattice. Let $\chi \in Fm$. We prove that $[\overline{\chi}] = \{\overline{\varphi} \in Fm^* : \overline{\chi} \leq \overline{\varphi}\} = \{\overline{\varphi} \in Fm^* : \chi \vdash_{\mathcal{S}} \varphi\}$ is a distributive lattice. In order to prove that $[\overline{\chi}]$ is a lattice, it is only necessary to show that

there exists the meet in $[\overline{\chi}]$. Let $\overline{\varphi}, \overline{\psi} \in [\overline{\chi}]$. So $\chi \vdash_{\mathcal{S}} \varphi, \psi$. Let us to prove that $m^*(\overline{\varphi}, \overline{\psi}, \overline{\chi})$ is the meet of $\overline{\varphi}$ and $\overline{\psi}$ in $[\overline{\chi}]$. By (B1) and (B2) we have $m(\varphi, \psi, \chi) \vdash_{\mathcal{S}} \varphi \vee \chi \vdash_{\mathcal{S}} \varphi$ and $m(\varphi, \psi, \chi) \vdash_{\mathcal{S}} \psi \vee \chi \vdash_{\mathcal{S}} \psi$. Thus $m^*(\overline{\varphi}, \overline{\psi}, \overline{\chi}) \leq \overline{\varphi}, \overline{\psi}$. Let $\overline{\gamma} \in [\overline{\chi}]$ be such that $\overline{\gamma} \leq \overline{\varphi}, \overline{\psi}$. So $\chi \vdash_{\mathcal{S}} \gamma$ and $\gamma \vdash_{\mathcal{S}} \varphi, \psi$. Then $\gamma \vdash_{\mathcal{S}} \varphi \vee \chi, \psi \vee \chi$. By (B3) we obtain that $\gamma \vdash_{\mathcal{S}} m(\varphi, \psi, \chi)$, that is, $\overline{\gamma} \leq m^*(\overline{\varphi}, \overline{\psi}, \overline{\chi})$. Hence $m^*(\overline{\varphi}, \overline{\psi}, \overline{\chi}) = \overline{\varphi} \wedge_{\overline{\chi}} \overline{\psi}$. Then we conclude that $\langle Fm^*, m^* \rangle$ is a nearlattice. Now we show that condition (D2) of Proposition 2.8 holds for $\langle Fm^*, m^* \rangle$. Let $\varphi, \psi, \gamma, \chi \in Fm$. Since $\langle Fm^*, m^* \rangle$ is a nearlattice, it follows that $\overline{\varphi} \vee^* m^*(\overline{\psi}, \overline{\gamma}, \overline{\chi}) \leq m^*(\overline{\varphi} \vee^* \overline{\psi}, \overline{\varphi} \vee^* \overline{\gamma}, \overline{\chi})$. In order to prove the inverse inequality, we need to show that $m(\varphi \vee \psi, \varphi \vee \gamma, \chi) \vdash_{\mathcal{S}} \varphi \vee m(\psi, \gamma, \chi)$. By (B1), $\psi, \gamma \vdash_{\mathcal{S}} \psi \vee \chi, \gamma \vee \chi$ and from (B3) we obtain that $\psi, \gamma \vdash_{\mathcal{S}} m(\psi, \gamma, \chi)$. Thus $\psi, \gamma \vdash_{\mathcal{S}} \varphi \vee m(\psi, \gamma, \chi)$. By (B4), it follows that

$$(3.3) \quad m(\psi, \gamma, \varphi \vee m(\psi, \gamma, \chi)) \vdash_{\mathcal{S}} \varphi \vee m(\psi, \gamma, \chi).$$

By (B1) and (B3), we can deduce $\varphi \vee \psi \vee \chi \vdash_{\mathcal{S}} \varphi \vee \psi \vee m(\psi, \gamma, \chi)$ and $\varphi \vee \gamma \vee \chi \vdash_{\mathcal{S}} \varphi \vee \gamma \vee m(\psi, \gamma, \chi)$. Then, by (B1)-(B3) and (3.3), we have

$$\begin{aligned} m(\varphi \vee \psi, \varphi \vee \gamma, \chi) &\vdash_{\mathcal{S}} \varphi \vee \psi \vee \chi, \varphi \vee \gamma \vee \chi \\ &\vdash_{\mathcal{S}} \varphi \vee \psi \vee m(\psi, \gamma, \chi), \varphi \vee \gamma \vee m(\psi, \gamma, \chi) \\ &\vdash_{\mathcal{S}} \psi \vee \varphi \vee m(\psi, \gamma, \chi), \gamma \vee \varphi \vee m(\psi, \gamma, \chi) \\ &\vdash_{\mathcal{S}} m(\psi, \gamma, \varphi \vee m(\psi, \gamma, \chi)) \\ &\vdash_{\mathcal{S}} \varphi \vee m(\psi, \gamma, \chi). \end{aligned}$$

Hence we have proved that $\langle Fm^*, m^* \rangle$ is a distributive nearlattice. Finally, we prove that \mathcal{S} is DN-based relative to $\{Fm^*\}$ and m . Let $\varphi_1, \dots, \varphi_n, \varphi \in Fm$. From property (B4), (2) of Proposition 3.6 and since \mathcal{S} is selfextensional, it follows that

$$\begin{aligned} \varphi_1, \dots, \varphi_n \vdash_{\mathcal{S}} \varphi &\iff m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \vdash_{\mathcal{S}} \varphi \\ &\iff \overline{m(\varphi_1, \dots, \varphi_n, \varphi)} \leq \overline{\varphi} \iff \overline{m(\varphi_1, \dots, \varphi_n, \varphi)} = \overline{\varphi} \\ &\iff Fm^* \models m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \approx \varphi \\ &\iff (\forall h \in \text{Hom}(Fm, Fm^*)) (m^{n-1}(h\varphi_1, \dots, h\varphi_n, h\varphi) = h\varphi) \\ &\iff (\forall h \in \text{Hom}(Fm, Fm^*)) (m^{n-1}(h\varphi_1, \dots, h\varphi_n, h\varphi) \leq h\varphi). \end{aligned}$$

This completes the proof. \square

Our next aim is to prove that every selfextensional logic \mathcal{S} with a DN-term is in fact fully selfextensional and the class $\text{Alg}(\mathcal{S})$ is a variety. It should be noted by the previous theorem and Proposition 3.4 that every selfextensional logic \mathcal{S} with a DN-term is DN-based relative to its intrinsic variety $K_{\mathcal{S}}$.

Proposition 3.9. *Let \mathcal{S} be a DN-based logic relative to m . Then, for every algebra $A \in K_{\mathcal{S}}$, the non-empty \mathcal{S} -filters of A are exactly the filters of the $\{m\}$ -reduct distributive nearlattice $\langle A, m^A \rangle$, i.e., $\text{Fi}_{\mathcal{S}}(A) \setminus \{\emptyset\} = \text{Fi}(A)$.*

Proof. Let $A \in \mathbf{K}_{\mathcal{S}}$. Let $F \in \mathbf{Fi}(A)$. Let $\varphi_1, \dots, \varphi_n, \varphi \in Fm$ be such that $\varphi_1, \dots, \varphi_n \vdash_{\mathcal{S}} \varphi$ and let $h \in \mathbf{Hom}(Fm, A)$ be such that $h(\varphi_i) \in F$ for all $i = 1, \dots, n$. By (3.1), we have $m^{n-1}(h(\varphi_1), \dots, h(\varphi_n), h(\varphi)) \leq h(\varphi)$. Since F is a filter of the nearlattice A and $h(\varphi_1), \dots, h(\varphi_n) \in F$, it follows by Proposition 2.10 that $m^{n-1}(h(\varphi_1), \dots, h(\varphi_n), h(\varphi)) \in F$. Then $h(\varphi) \in F$. Hence $F \in \mathbf{Fi}_{\mathcal{S}}(A)$. Conversely, let now $F \in \mathbf{Fi}_{\mathcal{S}}(A)$ be non-empty and let us prove, using Proposition 2.10, that F is a filter of A . Let $a, b \in F$ and $c \in A$. By (B1) and (B3) we have, for variables x, y and z , that $x, y \vdash_{\mathcal{S}} x \vee z, y \vee z \vdash_{\mathcal{S}} m(x, y, z)$. By taking $h \in \mathbf{Hom}(Fm, A)$ such that $h(x) = a, h(y) = b$ and $h(z) = c$, we obtain that $h(x), h(y) \in F$ and hence $h(m(x, y, z)) \in F$, i.e., $m(a, b, c) \in F$. Therefore, $F \in \mathbf{Fi}(A)$. \square

Theorem 3.10. *Let \mathcal{S} be a DN-based logic. Then:*

- (1) $\mathbf{Alg}(\mathcal{S}) = \mathbf{K}_{\mathcal{S}}$;
- (2) $\mathbf{Alg}(\mathcal{S})$ is a variety;
- (3) \mathcal{S} is DN-based relative to $\mathbf{Alg}(\mathcal{S})$.

Proof. (1) We know (Lemma 2.3) that $\mathbf{Alg}(\mathcal{S}) \subseteq \mathbf{K}_{\mathcal{S}}$. Let $A \in \mathbf{K}_{\mathcal{S}}$. From Proposition 3.9 we can easily deduce that the g-matrix $\langle A, \mathbf{Fi}_{\mathcal{S}}(A) \rangle$ is a reduced g-model of \mathcal{S} and hence $A \in \mathbf{Alg}(\mathcal{S})$. Properties (2) and (3) are immediate consequences of (1). \square

Corollary 3.11. *If \mathcal{S} is a DN-based logic, then \mathcal{S} is fully selfextensional.*

Proof. Let us prove this using the characterization (2) of fully selfextensionality of Definition 2.4. Let $A \in \mathbf{Alg}(\mathcal{S})$. So $A \in \mathbf{K}_{\mathcal{S}}$. Then, by Proposition 3.9, it is easy check that $\Lambda_A(\mathbf{Fi}_{\mathcal{S}}(A)) = \mathbf{Id}_A$. Hence \mathcal{S} is fully selfextensional. \square

We have characterized the DN-based logics as the selfextensional logics with a DN-term. As happens in the setting of selfextensional logics with a conjunction [9], two different sentential logics \mathcal{S} and \mathcal{S}' can be DN-based relative to the same DN-based variety \mathbf{K} . The unique possible case for this is when one of them has theorems and the other has not. Now we will see under what conditions the uniqueness can be obtained.

Let \mathbf{K} be a DN-based variety relative to a ternary term m . Let us define the sentential logic $\mathcal{S}_{\mathbf{K}}$ as follows: let $\varphi_1, \dots, \varphi_n, \varphi \in Fm$,

$$(3.4) \quad \varphi_1, \dots, \varphi_n \vdash_{\mathcal{S}_{\mathbf{K}}} \varphi \iff (\forall A \in \mathbf{K})(\forall h \in \mathbf{Hom}(Fm, A)) \\ m^{n-1}(h\varphi_1, \dots, h\varphi_n, h\varphi) \leq h\varphi$$

and

$$(3.5) \quad \emptyset \vdash_{\mathcal{S}_{\mathbf{K}}} \varphi \iff (\forall A \in \mathbf{K})(\forall h \in \mathbf{Hom}(Fm, A)) \\ (\forall a \in A)(a \leq h\varphi).$$

Now for every $\Gamma \subseteq Fm$, $\Gamma \vdash_{\mathcal{S}_{\mathbf{K}}} \varphi$ if and only if there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathcal{S}_{\mathbf{K}}} \varphi$. Notice that if $\mathcal{S}_{\mathbf{K}}$ has a theorem, then for every algebra $A \in \mathbf{K}$ the $\{m\}$ -reduct nearlattice $\langle A, m^A \rangle$ has a top element. Moreover, since \mathbf{K} is

a variety, it follows that K is the unique DN-based variety for which \mathcal{S}_K is DN-based and hence, by Theorem 3.10, we have $K_{\mathcal{S}} = \text{Alg}(\mathcal{S}_K) = K$.

A sentential logic \mathcal{S} is said to be *non-pseudo-axiomatic* ([10]) if for every formula φ , φ is a theorem if and only if φ is derivable from every formula ($\psi \vdash_{\mathcal{S}} \varphi$ for all formula ψ), or equivalently if the intersection of all its non-empty theories is the set of theorems. The following proposition is an immediate consequence from Definitions (3.4) and (3.5) and thus we omit its proof.

Proposition 3.12. *Let \mathcal{L} be an algebraic language and m be a ternary term of \mathcal{L} . If K is a DN-based variety relative to m , then \mathcal{S}_K is the unique non-pseudo-axiomatic sentential logic which is DN-based relative to K and m , and moreover $K_{\mathcal{S}_K} = K$. If \mathcal{S} is a DN-based and non-pseudo-axiomatic logic relative to m , then $\mathcal{S}_{K_{\mathcal{S}}} = \mathcal{S}$.*

Hence, under the condition of non-pseudo-axiomatic, we obtain the following kind of uniqueness for DN-based logics: different non-pseudo-axiomatic logics must be DN-based relative to different DN-based varieties.

Now we show a 1-1 correspondence between the class of DN-based and non-pseudo-axiomatic logics and the class of subvarieties of the variety axiomatized by equations (P1)-(P8) and (D1).

A sentential logic \mathcal{S}' is said to be an *extension* of a sentential logic \mathcal{S} if and only if for every $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma \vdash_{\mathcal{S}'} \varphi$.

Lemma 3.13. *Let \mathcal{S} and \mathcal{S}' be DN-based relative to m and non-pseudo-axiomatic logics. Then, $\Lambda(\mathcal{S}) \subseteq \Lambda(\mathcal{S}')$ if and only if \mathcal{S}' is an extension of \mathcal{S} .*

Proof. It is immediate that if \mathcal{S}' is an extension of \mathcal{S} , then $\Lambda(\mathcal{S}) \subseteq \Lambda(\mathcal{S}')$. So, we need to prove the implication from left to right. Assume that $\Lambda(\mathcal{S}) \subseteq \Lambda(\mathcal{S}')$. By property (B4) and properties (3) and (4) of Proposition 3.6, it follows that

$$\begin{aligned} \varphi_1, \dots, \varphi_n \vdash_{\mathcal{S}} \varphi &\iff m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \vdash_{\mathcal{S}} \varphi \iff \\ m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \dashv\vdash_{\mathcal{S}} \varphi &\implies m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \dashv\vdash_{\mathcal{S}'} \varphi \iff \\ m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \vdash_{\mathcal{S}'} \varphi &\iff \varphi_1, \dots, \varphi_n \vdash_{\mathcal{S}'} \varphi. \end{aligned}$$

Now, if $\emptyset \vdash_{\mathcal{S}} \varphi$ then for every formula ψ , $\psi \vdash_{\mathcal{S}} \varphi$. By we have just proved, we obtain that for every formula ψ , $\psi \vdash_{\mathcal{S}'} \varphi$. Now, since \mathcal{S}' is non-pseudo-axiomatic, it follows that $\emptyset \vdash_{\mathcal{S}'} \varphi$. Hence, we have proved that \mathcal{S}' is an extension of \mathcal{S} . \square

Let \mathcal{L} be an algebraic language and m be a ternary term of \mathcal{L} . We set

- $\mathbb{S}_m(\mathcal{L}) := \{\mathcal{S} : \mathcal{S} \text{ is a DN-based and non-pseudo-axiomatic logic relative to } m \text{ over } \mathcal{L}\}$ and
- $\mathbb{K}_m(\mathcal{L}) := \{K : K \text{ is a subvariety of the variety over } \mathcal{L} \text{ axiomatized by the equations (P1)-(P8) and (D1) with regard to } m\}$.

We consider $\mathbb{S}_m(\mathcal{L})$ ordered by the extension order, i.e., $\mathcal{S} \leq \mathcal{S}'$ iff \mathcal{S}' is an extension of \mathcal{S} and $\mathbb{K}_m(\mathcal{L})$ ordered by the inclusion order. Now we are in condition to establish and prove the announced result above.

Theorem 3.14. *Let \mathcal{L} be an algebraic language and m be a ternary term of \mathcal{L} . Then the map $F: \mathbb{S}_m(\mathcal{L}) \rightarrow \mathbb{K}_m(\mathcal{L})$ defined by: $F(\mathcal{S}) = K_{\mathcal{S}}$, is a dual order isomorphism.*

Proof. By Proposition 3.4, we have that F is well defined and by Proposition 3.12 we obtain that F is an onto map. Let $\mathcal{S}, \mathcal{S}' \in \mathbb{S}_m(\mathcal{L})$. Then, by Lemma 3.13 and using that \mathcal{S} and \mathcal{S}' are selfextensional, we have

$$\begin{aligned} \mathcal{S} \leq \mathcal{S}' &\iff \Lambda(\mathcal{S}) \subseteq \Lambda(\mathcal{S}') \\ &\iff (\forall \varphi, \psi \in Fm)(K_{\mathcal{S}} \models \varphi \approx \psi \implies K_{\mathcal{S}'} \models \varphi \approx \psi) \\ &\iff K_{\mathcal{S}'} \subseteq K_{\mathcal{S}}. \end{aligned}$$

Therefore F is a dual order isomorphism. \square

4. TWO EXAMPLES

4.1. The logic of distributive nearlattices. It is defined in [6] a sentential logic \mathcal{S}_{dn} , by means of a Gentzen calculus, which can be considered as naturally associated with the variety of distributive nearlattices. In [6], the logic \mathcal{S}_{dn} is denoted by $\mathcal{S}_{\mathbb{DN}}$, but here $\mathcal{S}_{\mathbb{DN}}$ has a specific definition, see (3.4) and (3.5). Let us show that in fact \mathcal{S}_{dn} is the unique non-pseudo-axiomatic and DN-based logic relative to the variety of distributive nearlattices \mathbb{DN} . To this end, we need to introduce some basic notions of Gentzen calculus; we refer the reader to [4] and [6] for more information.

Let Fm be the algebra of formulas of a given algebraic similarity type \mathcal{L} . For our purpose, we will consider a *sequent* of type \mathcal{L} to be a pair $\langle \Gamma, \varphi \rangle$ where Γ is a (possible empty) finite set of formulas and φ is a formula. As usual, we write $\Gamma \triangleright \varphi$ instead of $\langle \Gamma, \varphi \rangle$. Let us denote by $\text{Seq}(\mathcal{L})$ the collection of all sequents. A *Gentzen-style rule* is a pair $\langle X, \Gamma \triangleright \varphi \rangle$ where X is a (possible empty) finite set of sequents and $\Gamma \triangleright \varphi$ is a sequent. As usual, we shall use the standard fraction notation for Gentzen-style rules:

$$(4.1) \quad \frac{\Gamma_0 \triangleright \varphi_0, \dots, \Gamma_{n-1} \triangleright \varphi_{n-1}}{\Gamma \triangleright \varphi}$$

A *substitution instance* of a Gentzen-style rule $\langle X, \Gamma \triangleright \varphi \rangle$ is a Gentzen-style rule of the form $\langle \sigma[X], \sigma[\Gamma] \triangleright \sigma(\varphi) \rangle$ for some substitution $\sigma \in \text{Hom}(Fm, Fm)$. A *Gentzen calculus* is a set of Gentzen-style rules. Given a Gentzen calculus \mathbf{G} , the notion of a formal proof can be defined as usual. That is, a *proof* in the Gentzen calculus \mathbf{G} from a set of sequents X is a finite sequence of sequents each one of whose elements is a substitution instance of a rule of \mathbf{G} or a sequent in X or is obtained by applying a substitution instance of a rule of \mathbf{G} to previous elements in the sequence. A sequent $\Gamma \triangleright \varphi$ is *derivable in \mathbf{G} from a set of sequents X* if there is a proof in \mathbf{G} from X whose last sequent in the proof is $\Gamma \triangleright \varphi$. We express this writing $X \vdash_{\mathbf{G}} \Gamma \triangleright \varphi$.

Definition 4.1. A *Gentzen system* is a pair $\mathcal{G} = \langle Fm, \vdash_{\mathcal{G}} \rangle$ where $\vdash_{\mathcal{G}}$ is a finitary closure operator on the set $\text{Seq}(\mathcal{L})$ that is substitution-invariant and which satisfies the following *structural rules*: for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$,

$$\text{(Axiom)} \frac{\emptyset}{\varphi \triangleright \varphi} \quad \text{(Weakening)} \frac{\Gamma \triangleright \varphi}{\Gamma, \psi \triangleright \varphi} \quad \text{(Cut)} \frac{\Gamma \triangleright \varphi \quad \Gamma, \varphi \triangleright \psi}{\Gamma \triangleright \psi}$$

We say that a Gentzen system $\mathcal{G} = \langle Fm, \vdash_{\mathcal{G}} \rangle$ has a *Gentzen-style rule* of type (4.1) or (4.1) is a *Gentzen-style rule of \mathcal{G}* if $\Gamma_0 \triangleright \varphi_0, \dots, \Gamma_{n-1} \triangleright \varphi_{n-1} \vdash_{\mathcal{G}} \Gamma \triangleright \varphi$ and we say that a sequent $\Gamma \triangleright \varphi$ is a *derivable sequent* of \mathcal{G} when $\emptyset \vdash_{\mathcal{G}} \Gamma \triangleright \varphi$.

Let \mathbf{G} be a Gentzen calculus with the structural rules of (Axiom), (Weakening) and (Cut). Hence, \mathbf{G} defines in a standard way the Gentzen system $\mathcal{G}_{\mathbf{G}} = \langle Fm, \vdash_{\mathbf{G}} \rangle$ (see [4, 11]).

Now let $\mathcal{L} = \{m\}$ with m a ternary connective.

Definition 4.2 ([6, Definition 4.2]). Let $\mathcal{G}_{\mathbb{DN}} = \langle Fm, \vdash_{\mathbb{DN}} \rangle$ be the Gentzen system defined by the following Gentzen-style rules: the structural rules (Axiom), (Weakening) and (Cut) and the following rules

$$\begin{aligned} (\vee \triangleright) & \frac{\varphi \triangleright \chi \quad \psi \triangleright \chi}{\varphi \vee \psi \triangleright \chi} & (\triangleright \vee) & \frac{\Gamma \triangleright \varphi \quad \Gamma \triangleright \psi}{\Gamma \triangleright \varphi \vee \psi} \\ (m \triangleright) & \frac{}{m(\varphi, \psi, \chi) \triangleright \varphi \vee \chi} \quad \frac{}{m(\varphi, \psi, \chi) \triangleright \psi \vee \chi} \\ (\triangleright m) & \frac{\Gamma \triangleright \varphi \vee \chi \quad \Gamma \triangleright \psi \vee \chi}{\Gamma \triangleright m(\varphi, \psi, \chi)} & (m^n \triangleright) & \frac{\varphi_1, \dots, \varphi_n \triangleright \varphi}{m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \triangleright \varphi} \end{aligned}$$

Now the sentential $\mathcal{S}_{dn} = \langle Fm, \vdash_{\mathbb{DN}} \rangle$ is defined as follows: for all $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash_{\mathbb{DN}} \varphi \iff \text{there is a finite } \Gamma_0 \subseteq \Gamma \text{ such that } \vdash_{\mathbb{DN}} \Gamma_0 \triangleright \varphi.$$

Theorem 4.3 ([6, Theorem 4.15]). *The sentential logic \mathcal{S}_{dn} has the following properties:*

- (1) $\text{Alg}(\mathcal{S}_{dn}) = \mathbb{DN}$;
- (2) for all $\varphi_1, \dots, \varphi_n, \varphi \in Fm$,

$$\varphi_1, \dots, \varphi_n \vdash_{\mathbb{DN}} \varphi \iff \mathbb{DN} \models m^{n-1}(\varphi_1, \dots, \varphi_n, \varphi) \approx \varphi.$$

- (3) for every $A \in \text{Alg}(\mathcal{S}_{dn})$, $\text{Fi}_{\mathcal{S}_{dn}}(A) = \text{Fi}(A) \cup \{\emptyset\}$;
- (4) \mathcal{S}_{dn} is fully selfextensional.

Therefore, by condition (2) of the previous theorem, we have that the logic \mathcal{S}_{dn} is DN-based relative to the variety \mathbb{DN} . Moreover, since \mathcal{S}_{dn} is non-pseudo-axiomatic, it follows by Proposition 3.12 that $\mathcal{S}_{dn} = \mathcal{S}_{\mathbb{DN}}$.

4.2. Modal distributive nearlattices. Modal operators on distributive nearlattices were introduced and studied in [1]. There the main tool to treat these operators was a topological duality for the category of distributive nearlattices.

Now, let us consider the algebraic language $\mathcal{L} = \{m, \Box, \top\}$ of type $(3,1,0)$.

Definition 4.4. An algebra $\langle A, m, \Box, 1 \rangle$ is said to be a \Box -modal distributive nearlattice if $\langle A, m, 1 \rangle$ is a distributive nearlattice with top element and the following conditions hold:

- (1) $\Box 1 = 1$;
- (2) for all $a, b \in A$ such that $a \wedge b$ there exists, $\Box(a \wedge b) = \Box a \wedge \Box b$.

We denote by $\Box\mathbb{DN}$ the collection of all \Box -modal distributive nearlattices. Let us show that in fact, $\Box\mathbb{DN}$ is a variety.

Proposition 4.5. Let $\langle A, m, \Box, 1 \rangle$ be an algebra such that $\langle A, m, 1 \rangle$ is a distributive nearlattice with top element and $\Box 1 = 1$. Then, $\langle A, m, \Box, 1 \rangle \in \Box\mathbb{DN}$ if and only if the following identity holds in $\langle A, m, \Box, 1 \rangle$:

$$(\Box) \quad \Box m(x, y, z) = m(\Box(x \vee z), \Box(y \vee z), \Box z).$$

Therefore, $\Box\mathbb{DN}$ is the variety defined by identities of distributive nearlattices with top element and the identities $\Box 1 = 1$ and (\Box) .

Proof. First assume that $\langle A, m, \Box, 1 \rangle \in \Box\mathbb{DN}$. Let $a, b, c \in A$. Since the operator \Box is order-preserving, it follows that $\Box c \leq \Box(a \vee c)$ and $\Box c \leq \Box(b \vee c)$. Then, we have

$$\begin{aligned} \Box m(a, b, c) &= \Box[(a \vee c) \wedge (b \vee c)] = \Box(a \vee c) \wedge \Box(b \vee c) \\ &= (\Box(a \vee c) \vee \Box c) \wedge (\Box(b \vee c) \vee \Box c) = m(\Box(a \vee c), \Box(b \vee c), \Box c). \end{aligned}$$

Hence, the identity (\Box) holds in A . Now, conversely, suppose that the identity (\Box) holds in A . We need to check that condition (2) in Definition 4.4 is satisfied. First we show that \Box is order-preserving. Let $a, b \in A$ be such that $a \leq b$. So $b = b \vee a = m(b, b, a)$. Then, we have

$$\Box b = \Box m(b, b, a) = m(\Box(b \vee a), \Box(b \vee a), \Box a) = m(\Box b, \Box b, \Box a) = \Box b \vee \Box a.$$

Thus we obtain that $\Box a \leq \Box b$. Now let $a, b \in A$ be such that $a \wedge b$ there exists. Since \Box is order-preserving, we have $\Box(a \wedge b) \leq \Box a, \Box b$. Hence

$$\begin{aligned} \Box(a \wedge b) &= \Box m(a, b, a \wedge b) = m(\Box a, \Box b, \Box(a \wedge b)) \\ &= (\Box a \vee \Box(a \wedge b)) \wedge (\Box b \vee \Box(a \wedge b)) = \Box a \wedge \Box b. \end{aligned}$$

Therefore $\langle A, m, \Box, 1 \rangle$ is a \Box -modal distributive nearlattice. \square

Consider the DN-based logic $\mathcal{S}_{\Box\mathbb{DN}}$ on the algebraic language $\mathcal{L} = \{m, \Box, \top\}$ defined by (3.4) and (3.5). We already know that $\mathcal{S}_{\Box\mathbb{DN}}$ satisfies properties (B1)-(B4). Moreover, it is straightforward to show directly that the following conditions hold:

$$(N) \quad \vdash_{\mathcal{S}_{\Box\mathbb{DN}}} \varphi \text{ implies } \vdash_{\mathcal{S}_{\Box\mathbb{DN}}} \Box \varphi;$$

$$\begin{aligned}
&(\Box m) \quad \Box m(\varphi, \psi, \chi) \vdash_{\mathcal{S}_{\Box\Box\Box}} m(\Box(\varphi \vee \chi), \Box(\psi \vee \chi), \Box\chi); \\
&(m\Box) \quad m(\Box(\varphi \vee \chi), \Box(\psi \vee \chi), \Box\chi) \vdash_{\mathcal{S}_{\Box\Box\Box}} \Box m(\varphi, \psi, \chi).
\end{aligned}$$

Moreover, notice that $K_{\mathcal{S}_{\Box\Box\Box}} = \text{Alg}(\mathcal{S}_{\Box\Box\Box}) = \Box\Box\Box$ (see on page 12). Now let us show that $\mathcal{S}_{\Box\Box\Box}$ is the weakest selfextensional non-pseudo-axiomatic logic satisfying conditions (B1)-(B4), (N), $(\Box m)$ and $(m\Box)$.

Proposition 4.6. *Let \mathcal{S} be a selfextensional non-pseudo-axiomatic logic satisfying conditions (B1)-(B4), (N), $(\Box m)$ and $(m\Box)$. Then, \mathcal{S} is an extension of $\mathcal{S}_{\Box\Box\Box}$.*

Proof. Let \mathcal{S} be a selfextensional non-pseudo-axiomatic logic satisfying conditions (B1)-(B4), (N), $(\Box m)$ and $(m\Box)$. Then, by Theorems 3.8 and 3.10, we have that \mathcal{S} is DN-based relative to $\text{Alg}(\mathcal{S}) = K_{\mathcal{S}}$. Since \mathcal{S} satisfies conditions (N), $(\Box m)$ and $(m\Box)$, it is straightforward to show that $K_{\mathcal{S}} \subseteq \Box\Box\Box$. Hence, since $K_{\mathcal{S}} \subseteq \Box\Box\Box = K_{\mathcal{S}_{\Box\Box\Box}}$ and by Theorem 3.14, we obtain that $\mathcal{S}_{\Box\Box\Box} \leq \mathcal{S}$. \square

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